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Optimal Control of Gortler Vortices in a High-Reynolds number Asymptotics Framework

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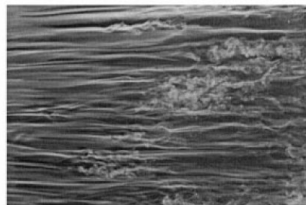
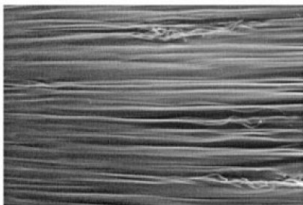
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Background

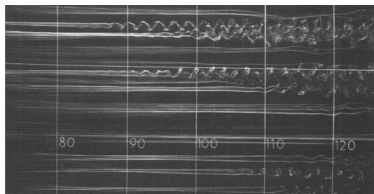
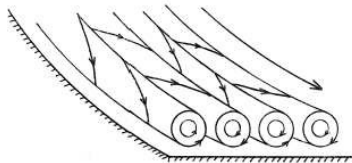
Klebanoff modes

-streamwise vortices in bypass transition



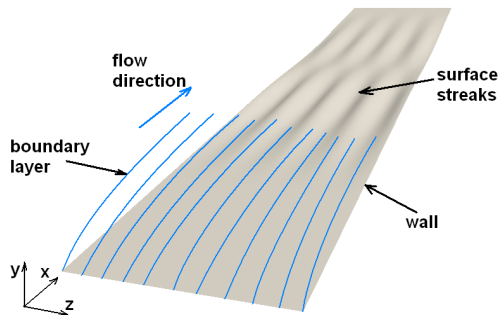
Gortler vortices

-centrifugal instabilities in boundary layers



Problem statement

Objective: We aim at deforming the wall to reduce the energy of Gortler vortices, and eventually to delay the secondary instabilities.



Practicality: future smart materials or actuators can be used to counteract boundary layer disturbances/streaks.

High Reynolds number asymptotics

We consider an incompressible boundary layer flow in a domain $\Omega \in \mathbb{R}^3$. The incompressible form of Navier Stokes equations:

$$\partial_t \mathbf{u}^* + (\mathbf{u}^* \cdot \nabla) \mathbf{u}^* = \nu \nabla^2 \mathbf{u}^* \quad (1)$$

$$\nabla \cdot \mathbf{u}^* = 0 \quad (2)$$

with initial condition:

$$\mathbf{u}^*(x, 0) = \mathbf{u}_0^*(x) \quad (3)$$

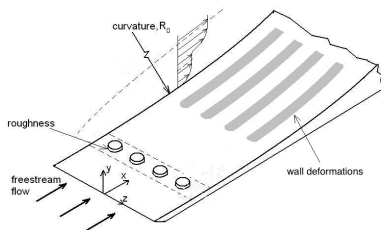
and appropriate boundary conditions:

$$\mathbf{u}^*(y, t) = \mathbf{u}_b^*(y, t), \quad y \in \partial\Omega, \quad t > 0 \quad (4)$$

High Reynolds number asymptotics

The wall is a concave surface, while the upstream perturbations are provided by localized roughness elements.

The spanwise length scale of the roughness row, Λ , is in the same order of magnitude as the local boundary-layer thickness $\delta^* \equiv x_0^* / \sqrt{R} = x_0^* \delta$.



The spatial coordinates are normalized by the span-wise length scale, Λ^* , as $(x, y, z) = (x^*, y^*, z^*) / \Lambda^*$. The velocity and pressure are normalized as

$$\tilde{u} = \frac{u^*}{U_\infty^*}, \tilde{v} = R_\Lambda \frac{v^*}{U_\infty^*}, \tilde{w} = R_\Lambda \frac{w^*}{U_\infty^*}, \tilde{p} = R_\Lambda^2 \frac{p^*}{\rho^* U_\infty^{*2}}, \quad (5)$$

where $R_\Lambda = U_\infty^* \Lambda^* / \nu^*$ is the Reynolds number based on Λ^* .

High Reynolds number asymptotics (*Cont'd*)

The velocity field $\{\tilde{u}, \tilde{v}, \tilde{w}\}$ and the pressure \tilde{p} are expanded like

$$\{\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}\} = \{u(X, y, z), \varepsilon v(X, y, z), \varepsilon w(X, y, z), \varepsilon^2 p(X, y, z)\} + \dots \quad (6)$$

where the small parameter $\varepsilon = 1/R_\Lambda$.

Upon substitution in the Navier-Stokes equations, the boundary region equations (BRE) are derived in the form

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (7)$$

$$u \frac{\partial u}{\partial X} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad (8)$$

$$u \frac{\partial v}{\partial X} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + G_\Lambda u^2 = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \quad (9)$$

$$u \frac{\partial w}{\partial X} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}, \quad (10)$$

where the effect of the wall curvature is contained in the term involving the global Görtler number $G_\Lambda = R_\Lambda^2/R_0$.

High Reynolds number asymptotics (*Cont'd*)

Prandtl transformation is applied in order to incorporate local changes in wall surface geometry defined through the function $\mathcal{F}(X, y)$.

$$Y = y - \mathcal{F}, \quad \hat{v} = v - \left(u \frac{\partial \mathcal{F}}{\partial X} + w \frac{\partial \mathcal{F}}{\partial z} \right).$$

Using the chain rule, gives the transformed BRE's:

$$\frac{\partial u}{\partial X} + \frac{\partial \hat{v}}{\partial Y} + \frac{\partial w}{\partial z} = 0, \quad (11)$$

$$u \frac{\partial u}{\partial X} + \hat{v} \frac{\partial u}{\partial Y} + w \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial Y^2} + \mathcal{D}^2 u - \frac{\partial^2 \mathcal{F}}{\partial z^2} \frac{\partial u}{\partial Y}, \quad (12)$$

$$u \frac{\partial v}{\partial X} + \hat{v} \frac{\partial v}{\partial Y} + w \frac{\partial v}{\partial z} + G_\Lambda u^2 = -\frac{\partial p}{\partial Y} + \frac{\partial^2 v}{\partial Y^2} + \mathcal{D}^2 v - \frac{\partial^2 \mathcal{F}}{\partial z^2} v_Y, \quad (13)$$

$$u \frac{\partial w}{\partial X} + \hat{v} \frac{\partial w}{\partial Y} + w \frac{\partial w}{\partial z} = -\mathcal{D} p + \frac{\partial^2 w}{\partial Y^2} + \mathcal{D}^2 w - \frac{\partial^2 \mathcal{F}}{\partial z^2} w_Y, \quad (14)$$

where the operator

$$\mathcal{D} = \left(\frac{\partial}{\partial z} - \frac{\partial \mathcal{F}}{\partial z} \frac{\partial}{\partial Y} \right)$$

was introduced for brevity.

These are the equations that will be used on our optimal control algorithm.

Optimal control problem

We write equations (7)-(10) in the compact and generic form

$$\mathcal{G}(\mathbf{q}, \psi) = 0, \quad (15)$$

with initial and boundary conditions

$$\mathbf{q}(0, Y, z) = \mathbf{q}_0(Y, z) \quad (16)$$

$$\mathbf{q}(X, 0, z) = \phi, \quad \lim_{Y \rightarrow \infty} \mathbf{q}(X, Y, z) = \mathbf{q}_B, \quad (17)$$

where:

- $\mathcal{G}()$ is a nonlinear differential operator,
- $\mathbf{q} = (u, v, w, p)$ is the state variable,
- ψ is the control variable that is part of the state equations (i.e., the functional $\mathcal{F}(X, z)$ describing the wall displacement),
- ϕ is the control variable from the boundary conditions (e.g., the transpiration velocity at the wall, v_w), and
- $\mathbf{q}_0(Y, z)$ and \mathbf{q}_B are given functions that specify initial conditions or certain boundary conditions, respectively.

We define an objective (or cost) functional as

$$\mathcal{J}(\mathbf{q}, \psi, \phi) = \mathcal{E}(\mathbf{q}) + \sigma_1 \left(\|\psi_X\|^{\beta_1} + \|\psi\|^{\beta_1} \right) + \sigma_2 \left(\|\phi_X\|^{\beta_2} + \|\phi\|^{\beta_2} \right), \quad (18)$$

where:

- $\mathcal{E}(\mathbf{q})$ is a specified target function to be minimized
- the second and the third terms on the right hand side are penalization terms,
- σ_i and β_i , $i = 1, 2$, are given constants.

We consider the Lagrangian

$$\boxed{\mathcal{L}(\mathbf{q}, \psi, \phi, \mathbf{q}^a) = \mathcal{J}(\mathbf{q}, \psi, \phi) - \langle \mathcal{G}(\mathbf{q}, \psi), \mathbf{q}^a \rangle}, \quad (19)$$

where \mathbf{q}^a is the vector of Lagrange multipliers (u^a, v^a, w^a, p^a), also known as the adjoint vector.

The Lagrange multipliers are introduced in order to transform the **constrained minimization** of $\mathcal{J}(\mathbf{q}, \psi, \phi)$ under the constraint $\mathcal{G}(\mathbf{q}, \psi) = 0$ into the **unconstrained minimization** of $\mathcal{L}(\mathbf{q}, \psi, \phi, \mathbf{q}^a)$.

Optimal control problem (*Cont'd*)

The unconstrained optimization problem can be formulated as:

Find the control variables ψ and ϕ , the state variables \mathbf{q} , and the adjoint variables \mathbf{q}^a such that the Lagrangian $\mathcal{L}(\mathbf{q}, \psi, \phi, \mathbf{q}^a)$ is a stationary function, that is

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \delta \mathbf{q} + \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \mathbf{q}^a} \delta \mathbf{q}^a = 0 \quad (20)$$

where

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a}} \delta \mathbf{a} = \frac{\mathcal{L}(\mathbf{a} + \epsilon \delta \mathbf{a}) - \mathcal{L}(\mathbf{a})}{\epsilon} \quad (21)$$

represents directional differentiation in the generic direction $\delta \mathbf{a}$.

The directional derivatives of the Lagrangian provides different sets of equations:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad \Rightarrow \quad \boxed{\mathcal{G}^a(\mathbf{q}^a, \psi) = 0} \quad (\text{adjoint equation}) \quad (22)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad \boxed{\mathcal{O}(\mathbf{q}^a, \mathbf{q}, \psi, \phi) = 0} \quad (\text{optimality condition}) \quad (23)$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}_a} = 0 \quad \Rightarrow \quad \boxed{\mathcal{G}(\mathbf{q}, \psi) = 0} \quad (\text{original state equation}) \quad (24)$$

Application to Gortler instabilities

Using integration by parts of the Lagrangian of BRE, we obtain the adjoint equations

$$\frac{\partial v^a}{\partial Y} + \mathcal{D}w^a = 0, \quad (25)$$

$$\begin{aligned} & -u \frac{\partial u^a}{\partial X} - \hat{v} \frac{\partial u^a}{\partial Y} - w \frac{\partial u^a}{\partial z} + u^a \frac{\partial u}{\partial X} + v^a \frac{\partial v}{\partial X} + w^a \frac{\partial w}{\partial X} - \frac{\partial \mathcal{F}}{\partial X} (u^a \frac{\partial u}{\partial Y} + v^a \frac{\partial v}{\partial Y} + w^a \frac{\partial w}{\partial Y}) \\ & + 2G_{\Lambda} u v^a - \frac{\partial p^a}{\partial X} + \frac{\partial \mathcal{F}}{\partial X} \frac{\partial p^a}{\partial Y} - \frac{\partial^2 u^a}{\partial Y^2} - \mathcal{D}^2 u^a - \frac{\partial^2 \mathcal{F}}{\partial z^2} \frac{\partial u^a}{\partial Y} = 0, \end{aligned} \quad (26)$$

$$-u \frac{\partial v^a}{\partial X} - \hat{v} \frac{\partial v^a}{\partial Y} - w \frac{\partial v^a}{\partial z} + u^a \frac{\partial v}{\partial Y} + v^a \frac{\partial v}{\partial Y} + w^a \frac{\partial w}{\partial Y} - \frac{\partial p^a}{\partial Y} - \frac{\partial^2 v^a}{\partial Y^2} - \mathcal{D}^2 v^a - \frac{\partial^2 \mathcal{F}}{\partial z^2} \frac{\partial v^a}{\partial Y} = 0, \quad (27)$$

$$\begin{aligned} & -u \frac{\partial w^a}{\partial X} - \hat{v} \frac{\partial w^a}{\partial Y} - w \frac{\partial w^a}{\partial z} + u^a \frac{\partial u}{\partial z} + v^a \frac{\partial v}{\partial z} + w^a \frac{\partial w}{\partial z} - \frac{\partial \mathcal{F}}{\partial z} (u^a \frac{\partial u}{\partial Y} + v^a \frac{\partial v}{\partial Y} + w^a \frac{\partial w}{\partial Y}) \\ & - \mathcal{D}p^a - \frac{\partial^2 w^a}{\partial Y^2} - \mathcal{D}^2 w^a - \frac{\partial^2 \mathcal{F}}{\partial z^2} \frac{\partial w^a}{\partial Y} = 0, \end{aligned} \quad (28)$$

satisfying the initial and boundary conditions

$$(u^a, v^a, w^a, p^a)|_{X=X_t} = (0, 0, 0, 0) \text{ in } \Omega, \quad (29)$$

$$(u^a, v^a, w^a)|_{\Gamma} = \begin{cases} (\alpha(\tau_w - \tau_0), 0, 0) & \text{for } X \in [X_{s0}, X_{s1}] \\ (0, 0, 0) & \text{otherwise} \end{cases} \quad (30)$$

$$(u^a, v^a, w^a, p^a)|_{Y \rightarrow \infty} = (0, 0, 0, 0) \quad (31)$$

Optimality condition for control based on wall transpiration

$$\boxed{\frac{\partial^2 v_w}{\partial X^2} - v_w = \frac{1}{\sigma_2} \left(p^a + \frac{\partial v^a}{\partial Y} \right)}, \quad (32)$$

satisfying the boundary conditions

$$\frac{\partial v_w}{\partial X}(X_0) = 0, \quad \frac{\partial v_w}{\partial X}(X_1) = 0, \quad (33)$$

Optimality condition for control based on wall deformations

$$\boxed{\sigma_1 \frac{\partial^2 \mathcal{F}}{\partial X^2} - 2 \frac{\partial u^a}{\partial z} \frac{\partial^2 u}{\partial Y^2} \frac{\partial \mathcal{F}}{\partial z} - \sigma_1 \mathcal{F} = \frac{\partial^2 u^a}{\partial z^2} \frac{\partial u}{\partial Y} - 2 \frac{\partial u^a}{\partial z} \frac{\partial^2 u}{\partial Y \partial z} - \frac{\partial p^a}{\partial X} \frac{\partial u}{\partial Y} - \frac{\partial p^a}{\partial z} \frac{\partial w}{\partial Y}} \quad (34)$$

satisfying the boundary conditions

$$\frac{\partial \mathcal{F}}{\partial X}(X_0) = 0, \quad \frac{\partial \mathcal{F}}{\partial X}(X_1) = 0. \quad (35)$$

Alternatively, the optimality condition (32) or (34) can be replaced by the steepest descent method, wherein the objective function is updated according to

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} - \alpha \frac{d\mathcal{J}^{(n)}}{d\mathcal{F}^{(n)}} \quad (36)$$

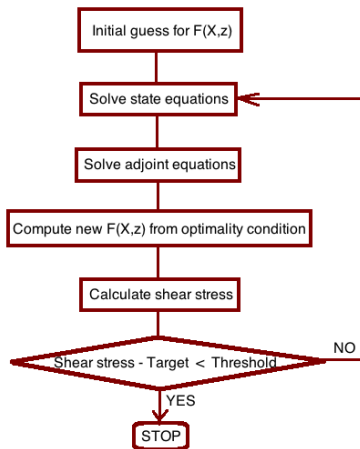
for control based on wall deformations, or

$$v_w^{(n+1)} = v_w^{(n)} - \alpha \frac{d\mathcal{J}^{(n)}}{dv_w^{(n)}} \quad (37)$$

for control based on wall transpiration, where n represent the iteration index, and α in this case is the descent parameter.

Application to Gortler instabilities (*Cont'd*)

The optimal control procedure:



Numerical methods

State equations: $\mathcal{G}(\mathbf{q}, \psi) = 0$ - nonlinear parabolic set of PDE

- marching in the streamwise direction
- pseudo-time relaxation in the y- and z-directions
- finite differencing

Adjoint equations: $\mathcal{G}^a(\mathbf{q}^a, \psi) = 0$ - linear parabolic set of PDE

- marching in the streamwise direction
- block-diagonal matrix inversion in the y- and z-directions

Optimal conditions:

- wall transpiration - linear elliptic ODE
- Jacobi relaxation method

- wall deformation - linear elliptic PDE
- Jacobi relaxation method

A row of roughness elements located at a fixed distance of 0.5 m from the leading edge is considered for three spanwise separation $\Lambda^* = 0.8$ cm.

The interval where the control is applied, $[X_0, X_1]$, and the interval where the sensor is placed, $[X_{s0}, X_{s1}]$, are the same.

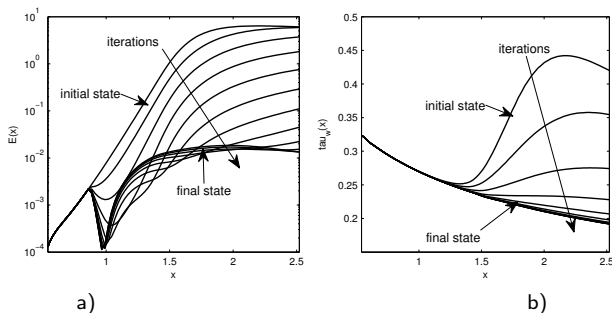
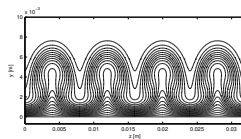
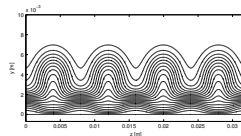
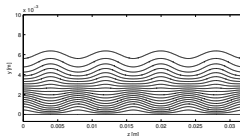
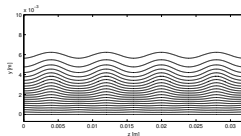


Figure: Typical convergence of the control algorithm: a) energy; c) spanwise averaged wall shear stress.

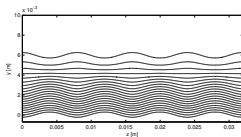
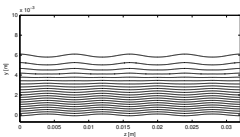
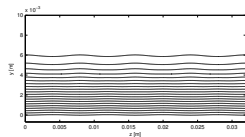


smooth surface

transpiration



deformation

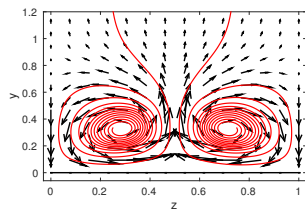


$x_c = 0.7$ m

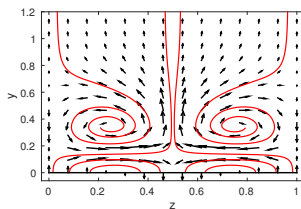
$x_c = 0.9$ m

$x_c = 1.1$ m

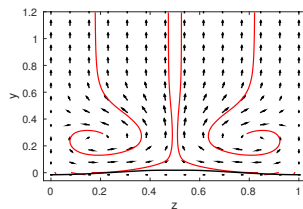
Figure: Streamwise velocity contours for spanwise separation of 0.8 cm.



uncontrolled flow



wall transpiration control



wall deformation control

Figure: Two-dimensional vector fields (v,w) and streamlines in a cross-plane (y,z) located in $X = 0.04$.

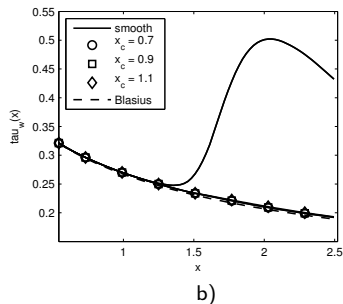
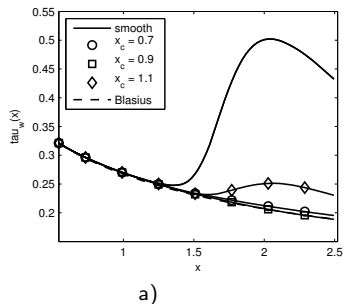
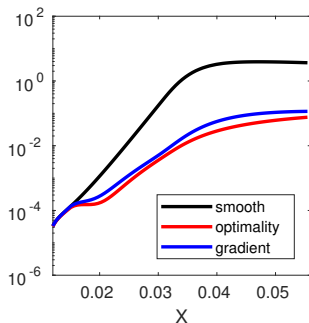


Figure: Spanwise averaged wall shear stress as a function of the streamwise direction for $\lambda = 0.8$ cm: a) control based on transpiration; b) control based on wall deformations.

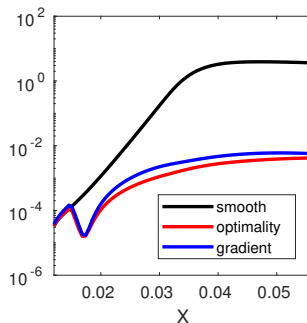
Gradient descent method versus optimality conditions

$$v_w^{(n+1)} = v_w^{(n)} - \alpha \frac{d\mathcal{J}^{(n)}}{dv_w^{(n)}}$$

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} - \alpha \frac{d\mathcal{J}^{(n)}}{d\mathcal{F}^{(n)}} \quad (38)$$



a)



b)

Figure: Spanwise averaged wall shear stress as a function of the streamwise direction: a) control based on transpiration; b) control based on wall deformations.

Conclusions:

We derived an optimal control strategy based on Lagrange multipliers to minimize the wall shear stress associated with Gortler instabilities.

The control was applied in a high-Reynolds number asymptotic framework where the Navier-Stokes equations have been reduced to the boundary region equations.

Prandtl transformation was used to account for wall deformations.

The results showed that the control algorithm is very efficient in reducing the energy and shear stress for given Gortler vortices that are excited by small roughness elements.

Future work will envision the application of the control strategy to unsteady flows and to other types of boundary layer streaks.

Thank you!